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Geometry of Adiabatic Changes. General Analysis

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The concept of adiabatic change and related notions are discussed. As a representative example, the evolution of a neutron spin in a precessing magnetic field is described briefly. The general setting of the equations for Dirac's evolution coefficients is discussed from the geometric phase point of view. An exactly solvable model of nutation is exhibited and the properties of the solutions are analyzed to reveal the holonomic structure of the problem. The corresponding expression for the geometric phase differs nontrivially from the corresponding expression in the well-known case of the precessing field. In addition, this geometric phase has an imaginary part which completes the picture of spin evolution in a nutation mode. The approach proposed for nutation is used to reexamine the twisted Landau–Zener problem.

. . . Thus $\gamma_n(C)$ is given by a circuit integral in parametric space and is independent of how the circuit is traversed (provided of course that this is slow enough for the adiabatic approximation to hold).

M. V. Berry

1. INTRODUCTION

The geometric phase is a subject of active study. The number of publications analyzing different mathematical aspects and experimental consequences of its existence, beginning with Berry's (1984) pioneering paper, is impressive. Moreover, recent investigations have developed further the general picture drawn by Berry, Simon, Hannay, Aharonov, and Anandan *et al.* (see Mukunda and Simon, 1993). Nevertheless, it is worthwhile returning and considering in greater detail the most typical case, that of the evolution of the neutron spin accompanying the movement of a magnetic field. This paper originated

To the memory of my father.

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with an unsuccessful attempt to apply existing statements of the geometric phase calculation in the case of essentially nonadiabatic spin (characteristic vector) evolution under an *arbitrary* and *finite* movement rate of the magnetic field (parametric vector).

Unfortunately, the scheme of adiabatic iteration developed in Berry (1987) cannot be accepted as consistent or applicable, and Berry's (1990) second consideration seems to have originated in a dissatisfaction with the previous analysis. The argument for a new consideration arises naturally at the level of intuition: for an arbitrary excursion, a complex generalization of the geometric phase seems inevitable.

It must be stressed that the notion of a complex geometric phase is not new and was well developed by Garrison and Wright (1988) for *dissipative* systems. We will not discuss this work because it represents an example of the standard way of generalizing on the basis of non-Hermitian Hamiltonians, which, as a consequence, usually leads from real parameters to their complex counterparts. Another approach leading to the notion of the complex geometric phase was developed by Alber and Marsden (1994) in connection with solitons.

Berry's second consideration appeals to Dykhne's calculation connected with integration of the quantum amplitude through the introduction of the complex time plane. Is the picture of a *geometric* amplitude drawn in this paper complete? Appropriate remarks can be found in Section 6.

On the other hand, an arbitrary excursion implies the possibility of a nonadiabatic evolution of the system. Is Aharonov and Ananadan's (1987) consideration of the nonadiabatic case, founded on the assumption of unitarity, the most general one? It will be shown in Section 4 that the latter, in particular, does not cover the case presented in Section 5.

2. ADIABATIC CHANGE

The conclusions leading to the notion of adiabatic change can be initiated with the following simple illustration. Let us take the hand of a watch. The rod about which the hand rotates can be taken as an axis in a manner which permits *comparatively* free turning of the hand.

Now, let us transport this construction along a plane, always keeping the rod vertical. We see that the initial and final positions of the hand, after an excursion along a closed path, coincide if the speed of the motion is slow enough. From the geometric point of view, this result points out that the *parallel* transport law was fulfilled locally by the watch hand.

Definition 1. The changes in a physical system are adiabatic if they are caused by a parametric vector moving slowly enough that the characteristic vectors of the physical system move in accordance with the parallel transport law.

Now, if we place our construction on the surface of a cone, we observe an angle between the initial and final positions of the watch hand even for a motion with very slow speed. This is the so-called *(an)holonomy* effect, caused by the nonlocal difference between the cone surface and a plane.

The value of this angle is equal to the cone angle and is obtained as a result of local conservation of the parallel transport law. One could easily generalize this result for a particular movement on the sphere figured here as a sequence of parallel movements along the corresponding tangential cones, and arrive at the *solid angle law*

$$\alpha_N = \Omega_N = (1 - \cos\theta)\Delta\phi$$

for movement crossing the north pole, and

$$\alpha_S = \Omega_S = (1 + \cos\theta)\Delta\phi$$

for movement crossing the south pole.

Furthermore, it can be easily shown that the solid angle law remains valid for the movement of the watch hand having an arbitrary configuration on the sphere and under an arbitrary radial deformation. Consequently, we can confirm that α_N and α_S are *topological invariants.* In the quantum setting, consideration passes to the Hilbert space with the corresponding notions of horizontal and vertical *lifts* for the wave functions (Bohm *et al.*, 1991).

Does this picture of adiabatic change hold in the case of the quantum evolution of neutron spin in a magnetic field? Is there a real difference between the terms "topological" and "geometrical" (phase), which are usually identified in the context of holonomy analysis?

The Cartesian representation of the Hamiltonian of a neutron spin in a magnetic field is

$$\begin{pmatrix} Z & X - iY \\ X + iY & -Z \end{pmatrix}$$

But this representation is not convenient for the consideration below. The natural map for investigating the holonomy effects is a spherical one. In particular, it is evident from the natural ad hoc separation of the variables for a slow movement: H, accounting for the dynamic phase, and the angular variables ϕ and θ, which, in principle, can generate holonomic phenomena. So let us rewrite and hereafter use the Hamiltonian in spherical representation

$$H\cdot\begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta \end{pmatrix}$$

Two corresponding solutions of Pauli's equation

$$i\hbar\dot{\Psi} = -2\mu\mathbf{H}\hat{\mathbf{s}}\Psi$$

for opposite projections of the spin can be obtained easily and read as follows:

$$\Psi^o_+(\theta, \phi) = e^{-i\omega_L t/2}\begin{pmatrix}\cos(\theta/2)\cdot e^{-i\phi} \\ \sin(\theta/2)\end{pmatrix} \tag{1}$$

$$\Psi^o_-(\theta, \phi) = e^{i\omega_L t/2}\begin{pmatrix}-\sin(\theta/2)\cdot e^{-i\phi} \\ \cos(\theta/2)\end{pmatrix} \tag{2}$$

where θ and ϕ are the respective polar and azimuthal angles of the quantization axis ($\mathbf{H}$) in the chosen coordinate frame, $\omega_L = 2|\mu|H/\hbar$ is the frequency of the Larmor precession, and μ is the magnetic momentum of the neutron, $H = |\mathbf{H}|$. The following must additionally be stressed: these often cited and dedicated solutions describe the neutron spin evolution in an arbitrarily oriented and homogeneous magnetic field. However, the problem of neutron spin evolution in a magnetic field is unique. After passing from the stationary problem (1), (2), one also has an exactly solvable problem for the case of a precessing magnetic field with the corresponding (time-dependent) Schrödinger equation (Landau and Lifshitz, 1977). Moreover, as will be shown in Section 5, we can point to an alternative to the precession mode that completes the picture of neutron spin evolution naturally and which also has an exact solution.

For the conclusions below, in addition to the definition of adiabatic changes, the following two notions need to be specified:

Definition 2. The weak nonadiabatic changes in a physical system are those caused by the movement of the vector parameter with a finite rate, but which conserve the parallel transport law for the characteristic vectors of the physical system.

Definition 3. The strong nonadiabatic changes in a physical system are those which are accompanied by violations of the characteristic vector parallel transport law and can be caused by infinitely slow movement of the external vector parameter.

We now present a more detailed consideration.

3. PRECESSION

By a precessing (or rotating) field configuration we mean the following specific time dependence of the magnetic field components:

$$\begin{aligned} H_x &= H\sin\theta\cos(\omega t + \phi) \\ H_y &= H\sin\theta\sin(\omega t + \phi) \\ H_z &= H\cos\theta \end{aligned} \tag{3}$$

where ω is the angular rate corresponding to a rotation around the z axis, with the strength H and polar angle θ constants.

The solutions of Pauli's equation for the precessing field case are well known and can be written as follows:

$$\Psi(t) = C_+\Psi_+(t) + C_-\Psi_-(t), \qquad |C_+|^2 + |C_-|^2 = 1 \tag{4}$$

$$\Psi_+(t) = e^{-i(\Lambda-\omega)t/2}\begin{pmatrix} \sqrt{\dfrac{\Lambda + \omega_L \cos\theta - \omega}{2\Lambda}}\, e^{-i(\omega t+\phi)} \\ \sqrt{\dfrac{\Lambda - \omega_L \cos\theta + \omega}{2\Lambda}} \end{pmatrix} \tag{5}$$

$$\Psi_-(t) = e^{i(\Lambda+\omega)t/2}\begin{pmatrix} -\sqrt{\dfrac{\Lambda - \omega_L \cos\theta + \omega}{2\Lambda}}\, e^{-i(\omega t+\phi)} \\ \sqrt{\dfrac{\Lambda + \omega_L \cos\theta - \omega}{2\Lambda}} \end{pmatrix} \tag{6}$$

where

$$\Lambda = \sqrt{(\omega - \omega_L \cos\theta)^2 + \omega_L^2 \sin^2\theta}$$

It must be stressed that the $\Psi_\pm$ are orthogonal:

$$(\Psi_+, \Psi_-) = 0$$

Under the substitutions

$$\cos\frac{\Theta}{2} = \sqrt{\frac{\Lambda + \omega_L \cos\theta - \omega}{2\Lambda}} \tag{7}$$

$$\Phi = \omega t + \phi \tag{8}$$

the spinors in the expressions above can be rewritten in a form similar to (1), (2):

$$\Psi_+(t) = e^{i\alpha_+}\tilde{\Psi}_+(t) = e^{-i(\Lambda-\omega)t/2}\begin{pmatrix} \cos(\Theta/2)\, e^{-i\Phi} \\ \sin(\Theta/2) \end{pmatrix} \tag{9}$$

$$\Psi_-(t) = e^{i\alpha_-}\tilde{\Psi}_-(t) = e^{i(\Lambda+\omega)t/2}\begin{pmatrix} -\sin(\Theta/2)\, e^{-i\Phi} \\ \cos(\Theta/2) \end{pmatrix} \tag{10}$$

These are the states with the following definite projection on the z axis:

$$(\Psi_\pm, s_z\Psi_\pm) = \pm\tfrac{1}{2}\cos\Theta \tag{11}$$

In the Aharonov and Anandan (1987) approach, as was shown particularly by Bodnarchuk *et al.* (1996), the phases of the exponents before the spinors in (9), (10) can be expressed in a surprising manner through Θ and Φ, too:

$$\alpha_+ = -\frac{\Lambda - \omega}{2} t = -\frac{\omega_L \cos(\theta - \Theta)}{2} t + \frac{(1 + \cos \Theta)\, \Delta\Phi}{2} \tag{12}$$

$$\alpha_- = \frac{\Lambda + \omega}{2} t = \frac{\omega_L \cos(\theta - \Theta)}{2} t + \frac{(1 - \cos \Theta)\, \Delta\Phi}{2} \tag{13}$$

where $\Delta\Phi = \Phi - \phi\ (= \omega t)$. As a result, we arrive at formulas showing that the values of the dynamic phases

$$\beta_\pm = \mp\frac{\omega_L \cos(\theta - \Theta)}{2} t \tag{14}$$

differ from $\propto \pm$ by the Aharonov–Anandan phases:

$$\frac{(1 \pm \cos \Theta)\, \Delta\Phi}{2} \tag{15}$$

The Aharonov–Anandan phases are equal and can also be obtained in the considered case directly from the known spinor parts through integration over the corresponding holonomy connections,

$$\gamma_\pm = \alpha_\pm - \beta_\pm = i \int_0^t \left(\tilde{\Psi}_\pm, \frac{\partial}{\partial \tau} \tilde{\Psi}_\pm\right) d\tau \tag{16}$$

These changes vanish when there is no precession ($\omega = 0$) and are equal to the Berry phases in the adiabatic mode ($\omega \to 0$)

$$\frac{(1 \pm \cos \theta)\, \Delta\Phi}{2} \tag{17}$$

For the components of the observable polarization vector, in the precession case after the cyclic evolution ($t = 2\pi/\omega$), and with the simplifying assumptions

$$(C_+, C_-) = (1/\sqrt{2}, i/\sqrt{2}), \qquad \phi = 0$$

we can obtain the following expressions:

$$P_x = \sin \Theta \sin(\alpha_- - \alpha_+) \tag{18}$$

$$P_y = \cos(\alpha_- - \alpha_+) \tag{19}$$

$$P_z = -\cos \Theta \sin(\alpha_- - \alpha_+) \tag{20}$$

where

$$\alpha_- - \alpha_+ = \frac{2\pi\omega_L \cos(\theta - \Theta)}{\omega} + \Omega(\Theta)$$

and

$$\Omega(\Theta) = 2\pi(1 - \cos \Theta)$$

is the solid angle traversed by the magnetic field.

These expressions show that if we subtract the Larmor (local) precession, the polarization of the neutron is a case of classic vector parallel transport under field precession with a finite angular rate ω: the solid angle law is conserved. So, in agreement with Definition 2, the neutron spin evolutions under magnetic field precession can be qualified as weak nonadiabatic changes.

4. GENERAL SETTING

The consideration in Section 3 of spin evolution in a precessing field was intended to illustrate the holonomic phenomena of Berry and Aharonov–Anandan on the level of an exactly solvable problem. But is it enough to know the precessing field case for solving the problem announced in Section 1—calculation of the geometric phase for an arbitrary (and with a finite rate) path of motion of a magnetic field on a Poincaré sphere? The first approach connected with this problem, as mentioned, is that of Berry (1987). It is not just a theoretical abstraction: one arrives at this question from the analysis of experiments as well (Korneev *et al.*, 1995).

Let us consider some conclusions that arise relating to the basic assumption of the Aharonov–Anandan approach:

Conclusion 1. To be exact, from the cyclicity on the level of the vector parameter

$$\mathbf{R}(T) = \mathbf{R}(0)$$

the corresponding quantum cyclic analog does not follow,

$$\Psi(T) = e^{i\alpha(T)}\Psi(0)$$

but instead,

$$\Psi(T) = \sum_m C_m(0) e^{i\alpha_m(T)} \Psi_m(0) \tag{21}$$

So, the classical cycle does not directly imply the corresponding quantum cycle.

Conclusion 2. Moreover, for an arbitrary moment of time t, the expression

$$e^{i\alpha(t)}\Psi(t)$$

cannot be accepted as the general form for the quantum system under nonadiabatic evolution. So the Aharonov and Anandan (1987) time integration is valid when the general formula

$$\Psi(t) = \sum_m C_m(0)e^{i\alpha_m(t)}\Psi_m(t) \tag{22}$$

is reduced, as in the case of precession, to the separate evolution of the partial (with fixed quantum number) basic states.

Let us consider the general setting. Dirac's standard substitution

$$\Psi(t) = \sum_m C_m(t)\Psi_m(t)$$

for the Schrödinger equation [the corresponding $\Psi(t)$ spectrum is assumed to be nondegenerate] gives a system of first-degree linear equations:

$$\dot{C}_n = \sum_m C_m A_{mn}$$

With the aim of understanding the case of neutron spin evolution in detail, we shall limit our consideration to a two-level system analysis. As a consequence, we obtain well-known standard expressions, i.e., a system of two first-degree linear equations

$$-i\dot{C}_1 = A(t)C_1(t) + B(t)C_2(t) \tag{23}$$

$$-i\dot{C}_2 = C(t)C_1(t) + D(t)C_2(t) \tag{24}$$

A, B, C, D are equal,

$$A(t) = i(\Psi_1, \dot{\Psi}_1) - \frac{(\Psi_1, \hat{H}\Psi_1)}{\hbar}, \qquad B(t) = i(\Psi_1, \dot{\Psi}_2) \tag{25}$$

$$C(t) = i(\Psi_2, \dot{\Psi}_1), \qquad D(t) = i(\Psi_2, \dot{\Psi}_2) - \frac{(\Psi_2, \hat{H}\Psi_2)}{\hbar} \tag{26}$$

where $i(\Psi_m, \dot{\Psi}_n)$ are the so-called coefficients of the holonomy connection induced by $\Psi_n(q, \mathbf{R}(t))$. As is known, we can separate the equations for C_1 and C_2 by passing to second-degree differential equations. They are easy to obtain:

$$B\ddot{C}_1 - (\dot{B} + i(A + D)B)\dot{C}_1 + (i(\dot{B}A - B\dot{A}) + B^2C)C_1 = 0 \quad (27)$$

$$D\ddot{C}_2 - (\dot{D} + i(C + B)D)\dot{C}_2 + (i(\dot{D}C - D\dot{C}) + D^2A)C_2 = 0 \quad (28)$$

Finally, with the exponential substitutions

$$\begin{aligned} C_s &= \exp(i\alpha_s) = \exp[i(\gamma_s + \beta_s)] \\ &= \exp\left[i\left(\int^t g_s\, d\tau - \int^t \omega_s\, d\tau\right)\right], \qquad s = 1, 2 \end{aligned}$$

the above equations can be rewritten in the conventional form

$$p_s\ddot{C}_s + q_s\dot{C}_s + r_sC_s = 0, \qquad s = 1, 2 \quad (29)$$

which leads to the following first-degree quadratic equation for g:

$$p_s(\dot{g}_s + ig_s^2) + (q_s - 2ip_s\omega_s)g_s - (p_s\dot{\omega}_s - ip_s\omega_s{}^2 + q_s\omega_s + ir_s) = 0 \quad (30)$$

In essence, these are the most general equations determining the geometric phases for two-level systems.

Further steps leading to the solution of these equations [or equations (27), (28)] depend on the behavior of the holonomic coefficients and the expressions above combined with the latter. In the best case they can be reduced to ones from a known list of common first- (or second-) degree differential equations (Kamke, 1959).

Nevertheless, it is possible to simplify the above problem as follows: evidently the wave function Ψ can be represented in a general form somewhat different from (22):

$$\Psi = e^{iu}(e^{iz}\Psi_1 + e^{-iz}\Psi_2) \quad (31)$$

eliminating the so-called global phase u. The usefulness of this form becomes evident after its substitution into (23), (24): we arrive at the following system of first-degree equations:

$$2i\dot{z} = A - D + Be^{-2iz} - Ce^{2iz} \quad (32)$$

$$2i\dot{u} = A + D + Be^{-2iz} + Ce^{2iz} \quad (33)$$

which are generally transcendent, but with a separate equation for one of the unknown phases z.

If we constrain our consideration, however, to observables which do not contain time derivatives (like the polarization vector), an easier scheme is possible: for the specified class of observables, it is not necessary to know the global phase e^{iu} and we can consider only the following partial solution:

$$\Psi = e^{iz}\Psi_1 + e^{-iz}\Psi_2$$

or the equivalent one

$$\Psi = \sin(z)\tilde{\Psi}_1 + \cos(z)\tilde{\Psi}_2 \tag{34}$$

In Section 5 we explore the representation (34) in a problem which can be viewed as an alternative to the precession problem considered in Section 3.

5. NUTATION

Let us assume that the time dependence of the magnetic field components has the following unusual form:

$$\begin{aligned} H_x &= H \sin(\omega t + \theta)\cos\phi \\ H_y &= H \sin(\omega t + \theta)\sin\phi \\ H_z &= H \cos(\omega t + \theta) \end{aligned} \tag{35}$$

where θ is the initial (axial) polar angle and ϕ is the azimuthal angle fixed during the magnetic field changes.

The system (35) is associated directly with the motion on sphere that is alternative to precession, i.e., *nutation.* We use this terminology.

Is it possible to solve Schrödinger's equation for neutron spin evolution with the time dependence given in (35)? Are there significant differences between neutron spin evolution in a nutating field and the well-known case of the precessing field? Here we show that this problem is solvable exactly and the mentioned difference is significant.

The substitution of (35) into Schrödinger's equation

$$i\hbar \frac{\partial}{\partial t}\Psi = |\mu|\mathbf{H}\hat{\sigma}\Psi$$

gives

$$\begin{pmatrix} \dot\psi^1 \\ \dot\psi^2 \end{pmatrix} = -\frac{i\omega_{\rm L}}{2}\begin{pmatrix} \cos(\omega t) & \sin(\omega t)e^{-i\Phi} \\ \sin(\omega t)\, e^{i\Phi} & -\cos(\omega t) \end{pmatrix}\begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} \tag{36}$$

where ω is the characteristic frequency of the nutation. For simplicity, we assume $\theta = 0$.

Let us try to find the solution in the form

$$\begin{aligned} \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} &= a(t)\Psi_1 + b(t)\Psi_2 \\ &= a(t)\begin{pmatrix} \cos(\omega t/2)\, e^{-i\Phi} \\ \sin(\omega t/2) \end{pmatrix} e^{-i\omega_L t/2} + b(t)\begin{pmatrix} -\sin(\omega t/2)\, e^{-i\Phi} \\ \cos(\omega t/2) \end{pmatrix} e^{i\omega_{\rm L} t/2} \end{aligned} \tag{37}$$

As a result, we obtain

$$\dot{a}(t)\begin{pmatrix}\cos(\omega t/2)\, e^{-i\Phi}\\ \sin(\omega t/2)\end{pmatrix}e^{-i\omega_L t/2} + a(t)\,\frac{\omega}{2}\begin{pmatrix}-\sin(\omega t/2)\, e^{-i\Phi}\\ \cos(\omega t/2)\end{pmatrix}e^{-i\omega_L t/2}$$

$$+\ \dot{b}(t)\begin{pmatrix}-\sin(\omega t/2)\, e^{-i\Phi}\\ \cos(\omega t/2)\end{pmatrix}e^{i\omega_L t/2} - b(t)\,\frac{\omega}{2}\begin{pmatrix}-\sin(\omega t/2)\, e^{-i\Phi}\\ \cos(\omega t/2)\end{pmatrix}e^{i\omega_L t/2} = 0$$

or

$$\dot{a}\Psi_1 + a\,\frac{\omega}{2}\,\Psi_2 e^{-i\omega_L t} + \dot{b}\Psi_2 - b\,\frac{\omega}{2}\,\Psi_1 e^{i\omega_L t} = 0 \tag{38}$$

Further, using the parametrization announced in Section 4,

$$a(t) = \sin z(t), \qquad b(t) = \cos z(t)$$

we obtain

$$\dot{z}(\cos z\Psi_1 - \sin z\Psi_2) = \frac{\omega}{2}\,(\cos z\Psi_1 e^{i\omega_L t} - \sin z\Psi_2 e^{-i\omega_L t})$$

After multiplying by $(\cos z^*\Psi_1^\dagger - \sin z^*\Psi_2^\dagger)$ we have

$$\dot{z}\cosh[2\,\mathrm{Im}(z)]$$

$$= \frac{\omega}{2}\,\{\cosh[2\,\mathrm{Im}(z)]\cos(\omega_L t) + i\cos[2\,\mathrm{Re}(z)]\sin(\omega_L t)\} \tag{39}$$

or

$$\frac{d\,\mathrm{Re}(z)}{d\tau} = \frac{\omega}{2}\cos(\omega_L \tau) \tag{40}$$

$$\frac{d\,\mathrm{Im}(z)}{dt} = \frac{\omega}{2}\,\frac{\cos[2\,\mathrm{Re}(z)]}{\cosh[2\,\mathrm{Im}(z)]}\sin(\omega_L \tau) \tag{41}$$

Let us consider the integration of $\mathrm{Re}(z)$:

$$\mathrm{Re}(z) = \frac{\omega}{2}\int_0^t \cos(\omega_L \tau)\, d\tau = \frac{\omega}{2\omega_L}\sin(\omega_L t) \tag{42}$$

This result can be written in another useful form:

$$\mathrm{Re}(z) = \frac{\omega t}{2} - \frac{1}{2}\int_0^t [1 - \cos(\omega_L \tau)]\, d(\omega\tau) \tag{43}$$

This expression equals the solid angle drawn by the unit vector in the direction of the nutation around the instantaneous position of the polarization vector projection on the local azimuthal ("horizontal") plane. As we will see below, the real part of z describes the quantum corrections to the classical parallel transport law in the meridian plane ("vertical" drift). Integration of the imaginary part gives

$$\mathrm{Im}(z) = \frac{1}{2}\,\mathrm{arcsinh}\int_0^t \cos[2\,\mathrm{Re}(z)]\,\sin(\omega_L \tau)\,d\tau \tag{44}$$

or

$$\mathrm{Im}(z) = \frac{1}{2}\,\mathrm{arcsinh}\int_0^t \cos\left[\frac{\omega}{\omega_L}\sin(\omega_L t)\right]\sin(\omega_L \tau)\,d\tau \tag{45}$$

The latter represents an integral which can be expressed through incomplete cylindrical functions (Whittaker and Watson, 1927; Agrest and Maksimov, 1965)

$$\epsilon_\nu(i\beta, z) = \frac{1}{\pi i}\int_0^{i\beta} e^{z\,\mathrm{sh}\,t - \nu t}\,dt = \frac{1}{\pi}\int_0^{\beta} e^{z\sin\theta - \nu\theta}\,d\theta$$

or Weber functions

$$B_\nu(\beta, z) = \frac{1}{\pi}\int_0^{\beta} \sin(\nu\theta - z\sin\theta)\,d\theta$$

As a result, we obtain

$$\begin{aligned}\mathrm{Im}(z) &= \frac{1}{2}\,\mathrm{arcsinh}\left\{\frac{\omega\pi}{4i\omega_L}\right.\\ &\quad\times\left[-\epsilon\left(i\omega_L t, -\frac{\omega}{\omega_L}\right) + \epsilon^*\left(i\omega_L t, -\frac{\omega}{\omega_L}\right)\right.\\ &\quad\left.\left. - \epsilon\left(i\omega_L t, \frac{\omega}{\omega_L}\right) + \epsilon^*\left(i\omega_L t, \frac{\omega}{\omega_L}\right)\right]\right\}\\ &= \frac{1}{2}\,\mathrm{arcsinh}\left\{\frac{\omega\pi}{2\omega_L}\left[B_1\left(\omega_L t, -\frac{\omega}{\omega_L}\right) + B_1\left(\omega_L t, \frac{\omega}{\omega_L}\right)\right]\right\}\end{aligned} \tag{46}$$

Let us consider the expressions for the components of the polarization vector P. Under the two simplifying assumptions $\phi = 0$ and $t = 2\pi/\omega_L$ it is easy to obtain the following remarkable expressions:

$$P_x = \frac{1}{\cosh(2\,\mathrm{Im}(z))} \sin\left[2\,\mathrm{Re}(z) - 2\pi \frac{\omega}{\omega_L}\right] \tag{47}$$

$$P_y = \tanh[2\,\mathrm{Im}(z)] \tag{48}$$

$$P_z = \frac{1}{\cosh[2\,\mathrm{Im}(z)]} \cos\left[2\,\mathrm{Re}(z) - 2\pi \frac{\omega}{\omega_L}\right] \tag{49}$$

So, the variable weight in (39) is a significant result. It is generated by the imaginary part of the geometric phase and cannot be ignored (calibrated) because it carries information about the spin evolution.

It is important that the quantum phases in the arguments above do not vanish for the elementary cyclic nutation—the excursion of the magnetic field along one of the meridians during the first half of the period, π/ω, and in the opposite direction for the second half of the period.

Conclusion 3 (Theorem). The paths on the Poincaré sphere for a magnetic field can be taken as a consequence of two alternative types of motion: precession and nutation. The corresponding evolutions for the polarization vector differ significantly: weak nonadiabatic changes in the case of precession and strong nonadiabatic changes in the case of nutation.

Let us apply this approach to the more complex problem discussed in Berry (1990).

6. TWISTED LANDAU–ZENER PROBLEM

This problem corresponds to the following magnetic field configuration:

$$\begin{aligned} H_x &= \Delta \cos[\Phi(t)] \\ H_y &= \Delta \sin[\Phi(t)] \\ H_z &= At \end{aligned} \tag{50}$$

The question arising in connection with this problem is the following: For $A \to 0$, this problem should transform into the precession problem discussed above. However, the expression for the geometric phase obtained in Berry (1990) has a singularity in this limit:

$$\Gamma_g = -\pi \frac{B\Delta^2}{A^2} \operatorname{sgn}(A) \to \infty$$

where $B = \ddot{\Phi}/2 = \text{const}$. This shows that the quasiclassical conclusions founded on the representation

$$\Gamma_g = -2\,\mathrm{Im} \int_0^{i\Delta/A} d\tau\, \dot{\Phi} \cos\theta$$

cannot be accepted as satisfactory and must be improved.

Let us search for the solution in the following form:

$$\begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} = \sin z(t) \begin{pmatrix} \cos(\Theta(t)/2)\, e^{-i\Phi(t)} \\ \sin(\Theta(t)/2) \end{pmatrix} e^{i(\beta_+ + \gamma_+)} + \cos z(t) \begin{pmatrix} -\sin(\Theta(t)/2)\, e^{-i\Phi(t)} \\ \cos(\Theta(t)/2) \end{pmatrix} e^{i(\beta_- + \gamma_-)} \tag{51}$$

where

$$\beta_\pm = \mp\frac{1}{2} \int_0^t \omega_L(\tau) \cos[\theta(\tau) - \Theta(\tau)]\, d\tau \tag{52}$$

$$\gamma_\pm = \frac{1}{2} \int_0^t [1 \pm \cos\Theta(\tau)]\, d\Phi \tag{53}$$

and

$$\omega_L(t) = 2\mu H(t)/\hbar, \qquad H(t) = \sqrt{\Delta^2 + A^2 t^2} \tag{54}$$

$$\theta(t) = \arctan(\Delta/At) \tag{55}$$

$$\cos\frac{\Theta(t)}{2} = \sqrt{\frac{\Lambda(t) + \omega_L(t)\cos\theta - \omega(t)}{2\Lambda(t)}} \tag{56}$$

$$\omega(t) = \frac{d\Phi(t)}{dt} \tag{57}$$

$\Lambda(t)$ is determined as in Section 3.

Steps similar to the ones in Section 5 give us the following expressions:

$$\mathrm{Re}(z) = \frac{1}{2} \int_0^t \cos[\Delta\alpha(\tau)]\, d\Theta \tag{58}$$

$$\mathrm{Im}(z) = \frac{1}{2} \operatorname{arcsinh} \int_0^t \cos[2\,\mathrm{Re}(z)] \sin[\Delta\alpha(\tau)]\, d\Theta \tag{59}$$

where $\Delta\alpha = \alpha_- - \alpha_+ = \beta_- - \beta_+ + \gamma_- - \gamma_+$.

These values vanish when $A \to 0$, $\dot{\Phi} \to 0$ and, as a result, we obtain the proper limit: the state corresponding to the evolution in the precessing field.

The most general expressions (58) and (59) do not reduce to known incomplete functions, but they can be simplified and reduced to the latter in two important limiting modes: the plane accelerated mode

$$A \to 0, \qquad \dot{\Phi} \neq 0$$

and the untwisted mode or axial lift

$$\Phi(t) = \text{const}$$

This problem, of course, deserves a more detailed description, but we confine ourselves here to the remarks above.

7. SUMMARY

Generally, the picture of quantum evolution differs from (quasi) classical parallel transport. In particular, as we have seen, the picture of neutron spin evolution naturally contains the nutation mode, and one cannot ignore the nutation mode in the geometric-phase iterative calculation in the general case, i.e., for arbitrary configuration of the magnetic field moving at a finite rate. Conceptually, we have to accept, for logical completeness, that Berry's anzatz (the existence of the holonomy connection) must be applied to the amplitudes and constants of normalization, too. As a consequence, the general consideration must be nonunitary. Generally, in a quantum setting the term "topological phase" is incorrect: the precise notion is definitely "geometric phase."

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